SOLUTION OF DIFFUSION-TYPE PROBLEMS FOR EXPANDING OR CONTRACTING REGIONS

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We consider the problem of solving an equation of the type

$$a\Delta u - \frac{\partial u}{\partial t} = f(x, y, z, t)$$

(where a is a constant and f is a given function) when the initial state is given and when the form of the surface bounding the region changes with time without losing its similarity characteristics. We show that by introducing new variables and a function we can reduce the above problem to that of solving an equation of the form

$$a\Delta v - R^2 \frac{\partial v}{\partial t} + \frac{R^3 R''}{4a} \rho^2 v = F$$

where $R = R(t)$ is a function defining the rate of displacement of the boundary surface, $R'' = d^2 R / dt^2$, ρ is the spherical or polar radius, F is a known function of time and coordinates and the boundary conditions for the function v are given at a surface similar to the boundary surface for u, but stationary. We show that, when $R = \sqrt{Mt^2 + Nt + P}$ where M, N and P are any constants (in particular we may have $R = Mt + N$ and $R = \sqrt{Mt + N}$), then the homogeneous equation for v (with $f = 0$) allows the separation of variables in rectangular, cylindrical or spherical coordinates. This in particular will yield general solutions of the problems for a plate, a rectangular parallelepiped, a cylinder of finite length, a sphere or a spherical shell e. a. which may expand or contract. In the case when the form of $R(t)$ differs from one shown above, the solution of the initial problem for u can be reduced to a relatively simple integral equation.

1. Investigation of the problems of the type indicated above can be reduced to solving an equation of the form

$$a\Delta u = \frac{\partial u}{\partial t} + f(x, y, z, t) \tag{1.1}$$

where a is a constant and $f(x, y, z, t)$ is a given function of time and coordinates, with the initial state

$$u|_{t=0} = F(x, y, z)$$

given, and with the boundaries of the region varying with time.

Such problems occur, as we know, in the theory of diffusion, heat conduction, mechanics of soil e. a. (see e. g. [1]).

We shall begin with a certain general transformation of (1.1). Let $R = R(t)$ be any function of time, continuous and possessing continuous first and second derivatives. Replacing x, y and z with new variables $\xi = x / R$, $\eta = y / R$ and $\zeta = z / R$, we obtain

$$a\left(\frac{\partial^2 u}{\partial \xi^2} + \frac{\partial^2 u}{\partial \eta^2} + \frac{\partial^2 u}{\partial \zeta^2}\right) + RR'\left(\xi \frac{\partial u}{\partial \xi} + \eta \frac{\partial u}{\partial \eta} + \zeta \frac{\partial u}{\partial \zeta}\right) - R^2 \frac{\partial u}{\partial t} =$$

$$= R^2 f(\xi R, \eta R, \zeta R, t), \quad R' = \frac{dR}{dt} \tag{1.2}$$

Here the derivative $\partial u / \partial t$ is taken with ξ, η and ζ constant. Assuming that ξ, η and ζ are certain rectangular coordinates and introducing the corresponding radius vector ρ connecting the origin with the point ξ, η and ζ, we can write (1.2) as follows:

$$a\Delta_{\xi,\eta,\zeta} u + RR'(\rho, \operatorname{grad} u) - R^2 \frac{\partial u}{\partial t} = R^2 f(\xi R, \eta R, \zeta R, t) \tag{1.3}$$

where the subscripts ξ, η and ζ accompanying the Laplacian indicate that it is taken in these variables (in the following these subscripts will be omitted).

Let us now replace u with v according to

$$u = qv \tag{1.4}$$

$$q = R^{n/2} \exp - RR'\rho^2 / 4a, \quad \rho^2 = \xi^2 + \eta^2 + \zeta^2, \tag{1.5}$$

If the problem in question is three-dimensional, we should put $n = 3$, $n = 2$ if the problem is two-dimensional and $n = 1$ if it is one-dimensional (plane). It can easily be shown that (1.2) or (1.3) in this case becomes

$$a\Delta v - R^3 \frac{\partial v}{\partial t} + \frac{R^3 R''}{4a} \rho^2 v = \frac{R^2 f(\xi R, \eta R, \zeta R, t)}{q} \tag{1.6}$$
$$R'' = \frac{d^2 R}{dt^2} \qquad \left(\Delta = \frac{\partial^2}{\partial \xi^2} + \frac{\partial^2}{\partial \eta^2} + \frac{\partial^2}{\partial \zeta^2}\right)$$

If in addition we introduce a new "time" variable τ defined by

$$\tau = \int \frac{dt}{R^2(t)} \tag{1.7}$$

then (1.6) becomes

$$a\Delta v - \frac{\partial v}{\partial \tau} + \frac{R^3 R''}{4a} \rho^2 v = \frac{R^2 f}{q} \tag{1.8}$$

which differs from the original equation only in the fact that the term $\frac{1}{4} R^3 R'' \rho^2 v / a$ appears in its left side (of course one of them is written in the x, y, z, t-variables, and the other in ξ, η, ξ, τ). The initial condition $u|_{t=0} = F(x, y, z)$ is transformed using (1.4) and (1.5) into corresponding initial condition

$$v\Big|_{\tau=\tau_0} = \frac{F(\xi R, \eta R, \xi R)}{q}\Big|_{t=0} =$$
$$= [R(0)]^{1/2 n} \exp\left(\frac{R(0) R'(0)}{4a} \rho^2\right) F[\xi R(0), \eta R(0), \zeta R(0)]$$
$$\tau_0 = \tau\,|_{t=0}.$$

When $R'' = 0$, i.e. when

$$R = At + B \tag{1.9}$$

where A and B are constants, the differences between the form of (1.1) and (1.8) disappear and they differ only in the right sides which are known for both equations (*).

From this it follows, in particular, that if a solution is sought of the equation

$$a\Delta v - \frac{\partial v}{\partial \tau} = \frac{R^2 f}{q} \qquad (R = At + B) \tag{1.10}$$

for some region whose shape does not change with time and where the initial state of the system is known and arbitrary (depending on the time and coordinates) values are assigned to the function v, then an analogous problem can be solved for (1.1) for a region of the same shape, but expanding or contracting uniformly with time and preserving similarity.

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*) For a homogeneous equation $a \, \Delta \, u = \partial u / \partial t$ we obtain at once a well known theorem stating that if the function $u = \varphi(x, y, z, t)$ is its solution, so is

$$u = t^{-n/2} \exp\left(-\frac{x^2 + y^2 + z^2}{4at}\right) \qquad \varphi\left(\frac{x}{At}, \frac{y}{At}, \frac{z}{At}, -\frac{1}{A^2 t}\right)$$

This follows directly from the correspondence shown above of the equations and initial conditions for both problems and from the fact, that to each point $\xi^\circ, \eta^\circ, \zeta^\circ$ of the boundary (stationary) of the region for the function v, there corresponds a point on the boundary whose coordinates are

$$x^\circ = \xi^\circ R = \xi^\circ\ (At + B), \quad y^\circ = \eta^\circ\ (At + B), \quad z^\circ = \zeta^\circ\ (At + B)$$

in the problem for u. In particular, in all the cases when the indicated boundary value problem for v is solved in those coordinate systems in which the homogeneous equation

$$a\,\Delta w - \partial w\ /\ \partial\tau = 0 \qquad (1.11)$$

corresponding to (1.10) allows the separation of variables (i.e. in the rectangular, cylindrical and spherical systems), the method yields directly a solution in known functions, of the corresponding problems for a uniformly expanding or contracting regions such as an infinite plate of constant thickness, a rectangular parallelepiped, a cylinder of finite length, a sphere or a spherical shell, etc.

This is the solution for the case when $R = At + B$. A more general case (*) when

$$R^3R'' = \text{const} = -\alpha \neq 0$$

is of particular interest. Here equation (1.6) becomes

$$a\Delta v - R^2\frac{\partial v}{\partial t} - \frac{\alpha}{4a}\rho^2 v = \frac{R^2 f}{q} \qquad (1.12)$$

The corresponding homogeneous equation is

$$a\Delta w - R^2\frac{\partial w}{\partial t} - \frac{\alpha}{4a}\rho^2 w = 0 \qquad (1.13)$$

and we can easily see that the separation of variables is possible in the rectangular, cylindrical and spherical coordinates. By virtue of this fact, all problems which admit such a solution for Eq. (1.10) in the above coordinates (in particular for a plate of uniform thickness, a cylinder of finite length, a sphere or a spherical shell, etc.) with stationary boundaries, can also be solved for (1.12).

This in turn implies that the corresponding boundary value problem for (1.1) can be solved completely for regions of the same shape, expanding or contracting according to a law which ensures that each point ξ°, η°, ζ° of the stationary boundary surface in the problem for v has a corresponding point $x^\circ = \xi^\circ R$, $y^\circ = -\eta^\circ R$, $z^\circ = \zeta^\circ R$ in the original problem for u. Here R satisfies the equation $R^3R'' = -\alpha$ and is given, in the general case, by

$$R = \sqrt{(At + B)^2 - \alpha\ /\ A^2} \qquad (1.14)$$

where A, B and α are arbitrary constants. As we said before, $R = At + B$ and $R = \sqrt{Mt + N}$ where M and N are different constants, represent particular cases of the above formula. The initial state of the system, i.e. the function $u\,|_{t=0} = F(x,\ y,\ z)$ and the values of u (dependent on time and the coordinates of the relevant point on the surface) on the boundary surface moving according to one of the laws defined by (1.14), can be assigned arbitrarily.

2. We shall now consider in more detail the functions entering (1.13) during the separation of variables in the rectangular, cylindrical or spherical coordinates. We shall deal

*) $R = \sqrt{At + B}$, $\alpha = {}^1/_4\ A^2$ represents such a case. Obviously $R = At + B$ corresponds to $\alpha = 0$.

with the last two cases first, assuming for simplicity the radial or spherical symmetry. Equation (1.13) then becomes

$$\frac{a}{\rho^n}\frac{\partial}{\partial\rho}\left(\rho^n\frac{\partial w}{\partial\rho}\right)-\frac{\alpha}{4a}\rho^2 w = R^2\frac{\partial w}{\partial t} \qquad (n=1,\ 2) \tag{2.1}$$

where the values $n = 1$ and $n = 2$ refer to the cylindrical and spherical symmetry respectively. Assuming that

$$w = \lambda(\rho)\mu(t) \tag{2.2}$$

we find

$$\frac{a}{\rho^n}\frac{d}{d\rho}\left(\rho^n\frac{d\lambda}{d\rho}\right)+\frac{\beta-\alpha\rho^2}{4a}\lambda = 0 \tag{2.3}$$

$$\frac{d\mu}{dt} = -\frac{\beta}{4aR^2}\mu \tag{2.4}$$

where β is an arbitrary parameter appearing during the separation of variables. It can easily be shown that the same equations with $n = 0$ result, when the variable separation is performed on (1.13) in the rectangular coordinate system (not only in the one-dimensional, but also in general, three-dimensional case). Assuming therefore in the following that n can assume the values of zero, one and two, we can include the case of rectangular coordinates in our discussion.

Let us now consider (2.3). When $\alpha = 0$, its solution is well known and can be written in terms of trigonometric or Bessel functions [2]. When $\alpha \neq 0$, introduction in (2.3) of another variable $\theta = \rho^2$ yields

$$\theta\frac{d^2\lambda}{d\theta^2}+\frac{n+1}{2}\frac{d\lambda}{d\theta}+\frac{\beta-\alpha\theta}{16a^2}\lambda = 0 \tag{2.5}$$

Putting $\theta = \frac{2as}{\sqrt{\alpha}}$ and $\lambda = e^{-1/2 s}\varphi$ we find

$$s\frac{d^2\varphi}{ds^2}+\left(\frac{n+1}{2}-s\right)\frac{d\varphi}{ds}-\left(\frac{n+1}{4}-\frac{\beta}{8a\sqrt{\alpha}}\right)\varphi = 0 \tag{2.6}$$

which is the equation of the degenerate hypergeometric function $F(\gamma,\ \delta,\ s)$ whose parameters are

$$\gamma = \frac{1}{4}\left(n+1-\frac{\beta}{2a\sqrt{\alpha}}\right), \qquad \delta = \frac{n+1}{2}$$

Thus we see that in the present case the particular solutions $\lambda(\rho)$ are expressed in terms of known and extensively tabulated functions, and the same applies to solutions of boundary value problems.

3. When the form of R differs from that given by (1.14) and the product R^3R'' is not constant, solutions of the above boundary value problems for (1.1) can be reduced to relatively simple integral equations for the corresponding functions of v. To do this, we shall assume that the third term in the left side of (1.8) is known, transfer it to the right side of the equation and express the values of v inside the region in terms of its boundary values (obtained from the boundary values of u given by (1.4) and (1.5)) and of the right side, using the Green functions. This gives the required integral equations.

4. We shall, in addition, mention the case when $R = \sqrt{Mt + N}$ where M and N are constants. Here $R^3R'' = -(1/2 M)^2 = \text{const}$, therefore this corresponds to the case already discussed. Since we also have $RR' = 1/2 M = \text{const}$, the problem simplifies because the variables can now be separated in Eq. (1.3) (more exactly, in the equation obtained from (1.3) for $f = 0$) and there is no need to pass from u to v. Obviously, when R varies with time according to such a law, the boundary value problem for the original equation (1.1) can be solved for the second order conditions, when only the normal derivative of u is given at the moving boundary. Solution is also possible for certain, more complicated (combined) formulations of the boundary conditions.

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PERIODIC SOLUTIONS OF CERTAIN NONLINEAR AUTONOMOUS SYSTEMS

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The structure of the period of a parametric solution of a certain nonlinear autonomous system (which is in a sense a generalization of Liapunov systems) is investigated. The existence of a periodic solution is due to the existence of the necessary number of first integrals. Formulas for approximate calculation of the period are derived for cases where such a solution can be said to exist. The results can be applied to the study of periodic solutions of systems close to that analyzed here under principal-resonance conditions (in the sense of Malkin).

1. Formulation of the problem. We consider the system

$$dx_i / dt = a_{i1}x_1 + \ldots + a_{in}x_n + X_i(x_1, \ldots, x_n) \quad (i = 1, \ldots, n) \tag{1.1}$$

where a_{ij} are constants and X_i are analytic nonlinear functions of the variables $x_1, \ldots, x_n$.

Let us assume that Eq.

$$|a_{ij} - \delta_{ij}\rho| = 0 \tag{1.2}$$

has l zero roots associated with l groups of solutions, two roots $\pm\lambda\sqrt{-1}$, and no roots which are multiples of $\pm\lambda\sqrt{-1}$.

Applying a linear nonsingular transformation with constant coefficients, we transform system (1.1) into [1 and 2]

$$du_j / dt = U_j, \quad dy / dt = -\lambda z + Y, \quad dz / dt = \lambda y + Z$$

$$dv_i / dt = b_{i1}v_1 + \ldots + b_{im}v_m + V_i \tag{1.3}$$

$$(j = 1, \ldots, l; \ i = 1, \ldots, m, \ m + l + 2 = n)$$

where U_j, Y, Z, V_i are analytic nonlinear functions of the variables $u_1, \ldots, u_i$, y, z, $v_1, \ldots, v_m$, and where the constants b_{ij} are such that there are no zero roots or multiples of $\pm\lambda\sqrt{-1}$ among the roots of the equation $|b_{ij} - \delta_{ij}\rho| = 0$.

Let us assume that system (1.3) has $l + 1$ analytic first integrals

$$M_j(u) + M_j^{(1)}(u, y, z, v) = C_j \quad (j = 1, \ldots, l) \tag{1.4}$$

$$y^2 + z^2 + \psi(u, y, z, v) = C_{i+1} \tag{1.5}$$

where M_j are linear independent forms of the variables $u_1, \ldots, u_i$; $M_j^{(1)}$, ψ are